5 Measures and integrals on product spaces

5.1 The Product of measures

Definition 5.1. Let S, T be sets and $\mathcal{M} \subseteq \mathfrak{P}(S)$, $\mathcal{N} \subseteq \mathfrak{P}(T)$ be algebras of subsets. For $(A, B) \in \mathcal{M} \times \mathcal{N}$ we view $A \times B$ as a subset of $S \times T$, called a *rectangle*. We denote the set of rectangles by $\mathcal{M} \times \mathcal{N} \subseteq \mathfrak{P}(S \times T)$. Then, $\mathcal{M} \square \mathcal{N} \subseteq \mathfrak{P}(S \times T)$ denotes the algebra generated by the set of rectangles. We also call this the *product algebra*. Similarly, $\mathcal{M} \boxtimes \mathcal{N}$ denotes the σ-algebra generated by $\mathcal{M} \square \mathcal{N}$ which we call the *product σ-algebra*.

Proposition 5.2. *$\mathcal{M} \square \mathcal{N}$ consists of the finite disjoint union of elements of $\mathcal{M} \times \mathcal{N}$.*

Proof. **Exercise.** □

Proposition 5.3. *Let $\mathcal{M}'$, $\mathcal{N}'$ be the σ-algebras generated by $\mathcal{M}$ and $\mathcal{N}$ respectively. Then,*

$$\mathcal{N} \boxtimes \mathcal{M} = \mathcal{N}' \boxtimes \mathcal{M}'.$$

Proof. **Exercise.** □

Lemma 5.4. *Let $(S, \mathcal{M})$, $(T, \mathcal{N})$ be measurable spaces. Let $U \in \mathcal{M} \boxtimes \mathcal{N}$ and $p \in S$. Set $U_p := \{q \in T : (p, q) \in U\} \subseteq T$. Then, $U_p \in \mathcal{N}$.*

Proof. Let $\mathcal{A}$ denote the set of subsets $V \subseteq S \times T$ such that $V \in \mathcal{M} \boxtimes \mathcal{N}$ and $V_p \in \mathcal{N}$. Let $(A, B) \in \mathcal{M} \times \mathcal{N}$. Then the rectangle $A \times B$ is in $\mathcal{A}$ since $(A \times B)_p = B$ if $p \in A$ and $(A \times B)_p = \emptyset$ otherwise. Thus, all rectangles are in $\mathcal{A}$. Moreover, $\mathcal{A}$ is an algebra: Clearly $\emptyset \in \mathcal{A}$. Also, if $V \in \mathcal{A}$, then $\neg V \in A$ since $(\neg V)_p = \neg(V_p)$. Similarly, for $A, B \in \mathcal{A}$ we have $(A \cap B)_p = A_p \cap B_p$. So, $\mathcal{M} \square \mathcal{N} \subseteq \mathcal{A}$. But $\mathcal{A}$ is even a σ-algebra: Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of elements of $\mathcal{A}$. Then, $(\bigcup_{n \in \mathbb{N}} A_n)_p = \bigcup_{n \in \mathbb{N}} (A_n)_p$. Thus, $\mathcal{M} \boxtimes \mathcal{N} \subseteq \mathcal{A}$. But $\mathcal{A} \subseteq \mathcal{M} \boxtimes \mathcal{N}$ by construction. □

Lemma 5.5. *Let $(S, \mathcal{M})$, $(T, \mathcal{N})$, $(U, \mathcal{A})$ be measurable spaces and $f : S \times T \to U$ a measurable map, where $S \times T$ is equipped with the product σ-algebra $\mathcal{M} \boxtimes \mathcal{N}$. For $p \in S$ denote by $f_p : T \to U$ the map $f_p(q) := f(p, q)$. Then, f_p is measurable for all $p \in S$.*

Proof. Let $V \in \mathcal{A}$. Then, $f_p^{-1}(V) = (f^{-1}(V))_p$, using the notation of Lemma 5.4. But by that same Lemma, $(f^{-1}(V))_p \in \mathcal{N}$. □

Theorem 5.6. *Let $(S, \mathcal{M}, \mu)$ and $(T, \mathcal{N}, \nu)$ be measure spaces with σ-finite measures. Then, there exists a unique measure $\mu \boxtimes \nu$ on the measurable space $(S \times T, \mathcal{M} \boxtimes \mathcal{N})$ such that for sets of finite measure $A \in \mathcal{M}$ and $B \in \mathcal{N}$ we have*

$$(\mu \boxtimes \nu)(A \times B) = \mu(A)\nu(B).$$

Proof. At first we assume the measures to be finite. It is clear from Proposition 5.2 that $\mu\boxtimes\nu$, if it exists, is uniquely determined on $\mathcal{M}\square\mathcal{N}$ by additivity. A priori it is not clear, however, if $\mu\boxtimes\nu$ can be well defined even merely on $\mathcal{M}\square\mathcal{N}$, since a given element of $\mathcal{M}\square\mathcal{N}$ can be presented as a disjoint union of rectangles in different ways. For $U\in\mathcal{M}\square\mathcal{N}$ define $\alpha_U : S\to\mathbb{R}^+_0$ by $\alpha_U(p) := \nu(U_p)$. If $U = A\times B$ is a rectangle, we have $\alpha_U(p) = \chi_A(p)\nu(B)$ for $p\in S$. In particular, α_U is integrable on S and we have

$$\mu(A)\nu(B) = \int_S \alpha_U \,\mathrm{d}\mu.$$

For U a finite disjoint union of rectangles the function α_U is simply the sum of the corresponding functions for the individual rectangles and is thus integrable on S. In particular, we must have

$$(\mu\boxtimes\nu)(U) = \int_S \alpha_U \,\mathrm{d}\mu,$$

incidentally showing that $\mu\boxtimes\nu$ is well defined on $\mathcal{M}\square\mathcal{N}$.

We proceed to show that $\mu\boxtimes\nu$ is countably additive on $\mathcal{M}\square\mathcal{N}$. Let $\{U_n\}_{n\in\mathbb{N}}$ be an increasing sequence of subsets of $\mathcal{M}\square\mathcal{N}$ such that $U := \bigcup_{n\in\mathbb{N}} U_n \in\mathcal{M}\square\mathcal{N}$. Then, $\{\alpha_{U_n}\}_{n\in\mathbb{N}}$ is an increasing sequence of integrable functions on S such that

$$\int_S \alpha_{U_n}\,\mathrm{d}\mu \le \int_S \alpha_U \,\mathrm{d}\mu = (\mu\boxtimes\nu)(U) \quad \forall n\in\mathbb{N}.$$

Hence we can apply the Monotone Convergence Theorem 3.29. Since α_{U_n} converges pointwise to α_U we must have

$$\lim_{n\to\infty}\int_S \alpha_{U_n}\,\mathrm{d}\mu = \int_S \alpha_U \,\mathrm{d}\mu.$$

That is, $\lim_{n\to\infty}(\mu\boxtimes\nu)(U_n) = (\mu\boxtimes\nu)(U)$, implying countable additivity. It is now guaranteed by Hahn's Theorem 2.35 and Proposition 2.36 that $\mu\boxtimes\nu$ extends to a measure on $\mathcal{M}\boxtimes\mathcal{N}$, and uniquely so.

It remains to consider the case of σ-finite measures. **Exercise.** $\square$

Exercise 31. Show whether the operation of taking the product measure is associative.

Exercise 32. Show that the Lebesgue measure on $\mathbb{R}^{n+m}$ is the product measure of the Lesbegue measures on $\mathbb{R}^n$ and $\mathbb{R}^m$.

In the following we denote the completion of a σ-algebra $\mathcal{A}$ with respect to a given measure by $\mathcal{A}^*$.

Lemma 5.7. *Let $(S,\mathcal{M},\mu)$ and $(T,\mathcal{N},\nu)$ be measure spaces with σ-finite complete measures. Let $Z\in(\mathcal{M}\boxtimes\mathcal{N})^*$ of measure 0. Then, for almost all $p\in S$ we have $\nu(Z_p) = 0$.*

Proof. We consider first the case that the measures are finite. For all $n \in \mathbb{N}$ define $Y_n := \{p \in S : \nu(Z_p) \geq 1/n\}$. Now fix $n \in \mathbb{N}$ and $j \in \mathbb{N}$. Since the algebra $\mathcal{N}\square\mathcal{N}$ generates the σ-algebra $\mathcal{N} \boxtimes \mathcal{M}$, Theorem 2.35, implies that there is a sequence of disjoint rectangles $\{A_{j,k} \times B_{j,k}\}_{k\in\mathbb{N}}$ such that $Z \subseteq R_j$ and $(\mu \boxtimes \nu)(R_j) < 1/(nj)$, where $R_j := \bigcup_{k=1}^{\infty}(A_{j,k} \times B_{j,k})$. Define now $X_j := \{p \in S : \nu((R_j)_p) \geq 1/n\}$. Obviously, $Y_n \subseteq X_j$. Moreover, X_j is measurable since $p \mapsto \nu((R_j)_p) = \sum_{k=1}^{\infty} \chi_{A_{j,k}}(p)\nu(B_{j,k})$ is measurable, being a pointwise limit of measurable functions (Theorem 2.19). We have then,

$$
\begin{aligned}
(\mu \boxtimes \nu)(R_n) &= \sum_{k=1}^{\infty} \mu(A_{j,k})\nu(B_{j,k}) = \sum_{k=1}^{\infty} \int_S \chi_{A_{n,k}}(p)\nu(B_{j,k})\,\mathrm{d}\mu(p) \\
&= \int_S \sum_{k=1}^{\infty} \chi_{A_{n,k}}(p)\nu(B_{j,k})\,\mathrm{d}\mu(p) = \int_S \nu((R_j)_p)\,\mathrm{d}\mu(p) \\
&\geq \int_{X_n} \nu((R_j)_p)\,\mathrm{d}\mu(p) \geq \int_{X_j} \frac{1}{n}\,\mathrm{d}\mu = \frac{1}{n}\mu(X_n)
\end{aligned}
$$

(**Exercise.**Justify the interchange of sum and integral!) Thus we get the estimate $\mu(X_j) < 1/j$. Repeating the construction for all $j \in \mathbb{N}$ set $X := \bigcap_{j=1}^{\infty} X_j$. We then have $Y_n \subseteq X$, but $\mu(X) = 0$. Thus, since μ is complete, Y_n is measurable and has measure 0. This in turn implies that $Y := \{p \in S : \nu(Z_p) > 0\} = \bigcup_{n=1}^{\infty} Y_n$ has measure 0 as required. **Exercise.**Complete the proof for the σ-finite case! □

5.2 Fubini's Theorem

Lemma 5.8. *Let $(S, \mathcal{M}, \mu)$ and $(T, \mathcal{N}, \nu)$ be measure spaces with σ-finite measures. Let $A \times B \subseteq S \times T$ be a rectangle such that $0 < (\mu\boxtimes\nu)(A\times B) < \infty$. Then, $0 < \mu(A) < \infty$ and $0 < \nu(B) < \infty$.*

Proof. **Exercise.** □

Lemma 5.9. *Let $(S, \mathcal{M}, \mu)$ and $(T, \mathcal{N}, \nu)$ be measure spaces with σ-finite complete measures. Let $\{(\lambda_1, A_1, B_1), \ldots, (\lambda_n, A_n, B_n)\}$ be triples of elements of $\mathbb{K}, \mathcal{N}, \mathcal{M}$ respectively and such that $0 \leq \mu(A_i) < \infty$ and $0 \leq \nu(B_i) < \infty$. Define $g : S \times T \to \mathbb{K}$ by*

$$g(p, q) := \sum_{k=1}^{n} \lambda_k \chi_{A_k}(p)\chi_{B_k}(q).$$

Then, $g_p \in \mathcal{S}(T, \nu)$ for all $p \in S$ and

$$p \mapsto \int_T g_p\,\mathrm{d}\nu$$

defines a function in $\mathcal{S}(S,\mu)$ *satisfying*

$$\int_S \left(\int_T g_p \, \mathrm{d}\nu \right) \mathrm{d}\mu(p) = \int_{S\times T} g \, \mathrm{d}(\mu \boxtimes \nu).$$

Proof. **Exercise.** □

Theorem 5.10 (Fubini's Theorem, Part 1)**.** *Let* $(S,\mathcal{M},\mu)$ *and* $(T,\mathcal{N},\nu)$ *be measure spaces with* σ*-finite complete measures and* $f \in \mathcal{L}^1(S\times T, (\mathcal{M}\boxtimes\mathcal{N})^*, \mu\boxtimes\nu)$*. Then,* $f_p \in \mathcal{L}^1(T,\mathcal{N},\nu)$ *for almost all* $p \in S$ *and*

$$p \mapsto \int_T f_p \, \mathrm{d}\nu$$

defines almost everywhere a function in $\mathcal{L}^1(S,\mathcal{M},\mu)$ *satisfying*

$$\int_S \left(\int_T f_p \, \mathrm{d}\nu \right) \mathrm{d}\mu(p) = \int_{S\times T} f \, \mathrm{d}(\mu \boxtimes \nu).$$

Proof. By Proposition 3.23 there is a sequence $\{f_n\}_{n\in\mathbb{N}}$ of integrable simple functions, measurable with respect to $\mathcal{M}\square\mathcal{N}$, that converges to f in the $\|\cdot\|_1$-seminorm. Each function f_n can be written as a linear combination of characteristic functions on elements of $\mathcal{M}\square\mathcal{N}$ with finite measure. By modifying f_n if necessary, but without affecting convergence of the sequence we can also arrange that the supports of the characteristic functions all have non-negative measure. Due to Theorem 3.24, by replacing $\{f_n\}_{n\in\mathbb{N}}$ with a subsequence if necessary, we can ensure moreover pointwise convergence to f, except on a set N of measure zero. Taking into account Lemma 5.8 we notice that the functions f_n satisfy the conditions of Lemma 5.9.

By Lemma 5.7, there exists a subset $X \subseteq S$ with measure 0 such that $\nu(N_p) = 0$ if $p \notin X$. Fix for the moment $p \in S \setminus X$. Then, $\{(f_n)_p\}_{n\in\mathbb{N}}$ converges to f_p pointwise outside N_p. Moreover, since the $(f_n)_p$ are measurable with respect to $(T,\mathcal{N})$ by construction, so is f_p outside of N_p due to Theorem 2.19. But, Z_p has measure zero and $(T,\mathcal{N},\nu)$ is complete by assumption, so f_p is measurable everywhere.

Since $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy, we can restrict to a subsequence such that

$$\|f_l - f_k\|_1 < 2^{-2k} \quad \forall k \in \mathbb{N}, \forall l \geq k.$$

By applying Lemma 5.9 to $|f_l - f_p|$, we have for all $k \in \mathbb{N}$ and $l \geq k$,

$$\int_S \|(f_l)_p - (f_k)_p\|_{1,\nu} \, \mathrm{d}\mu(p) = \int_S \left(\int_T |(f_l)_p - (f_k)_p| \, \mathrm{d}\nu \right) \mathrm{d}\mu(p)$$
$$= \int_S \left(\int_T |f_l - f_k|_p \, \mathrm{d}\nu \right) \mathrm{d}\mu(p) = \int_{S\times T} |f_l - f_k| \, \mathrm{d}(\mu\boxtimes\nu) = \|f_l - f_k\|_1 < 2^{-2k}.$$

Now for $k \in \mathbb{N}$ set $Y_k \subseteq S$ to

$$Y_k := \left\{ p \in S : \|(f_l)_p - (f_k)_p\|_{1,\nu} \geq 2^{-k} \right\}.$$

Then, for all $k \in \mathbb{N}$,

$$2^{-k}\mu(Y_k) \leq \int_{Y_k} \|(f_{k+1})_p - (f_k)_p\|_{1,\nu} \mathrm{d}\mu(p)$$
$$\leq \int_S \|(f_{k+1})_p - (f_k)_p\|_{1,\nu} \mathrm{d}\mu(p) \leq 2^{-2k}.$$

This implies, $\mu(Y_k) \leq 2^{-k}$ for all $k \in \mathbb{N}$. Define now $Z_j := \bigcup_{k=j}^{\infty} Y_k$ for all $j \in \mathbb{N}$. Then, $\mu(Z_j) \leq 2^{1-j}$ for all $j \in \mathbb{N}$.

Fix $j \in \mathbb{N}$ and let $p \in S \setminus Z_j$. Then, for $k \geq j$ we have

$$\|(f_{k+1})_p - (f_k)_p\|_{1,\nu} < 2^{-k}.$$

This implies for $k \geq j$ and $l \geq k$,

$$\|(f_l)_p - (f_k)_p\|_{1,\nu} < 2^{1-k}.$$

In particular, $\{(f_n)_p\}_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to the $\|\cdot\|_{1,\nu}$-seminorm. Since j was arbitrary, this remains true for $p \in S \setminus Z$, where $Z := \bigcap_{j=1}^{\infty} Z_j$. Note that $\mu(Z) = 0$. Now let $p \in S \setminus (X \cup Z)$. Since $\{(f_n)_p\}_{n\in\mathbb{N}}$ converges to f_p pointwise almost everywhere, and f_p is measurable, Proposition 3.25 then implies that f_p is integrable and that $\{(f_n)_p\}_{n\in\mathbb{N}}$ converges to f_p in the $\|\cdot\|_{1,\nu}$-seminorm.

Now define

$$h_n : p \mapsto \int_T (f_n)_p \,\mathrm{d}\nu$$

By Lemma 5.9 this is an integrable simple map and by the previous arguments it converges pointwise outside of $X \cup Z$ to

$$h : p \mapsto \int_T (f)_p \,\mathrm{d}\nu.$$

Thus, h is measurable in $S \setminus (X \cup Z)$ by Theorem 2.19 and can be extended to a measurable function on all of S, for example by setting $h(p) = 0$ if $p \in X \cup Z$. On the other hand, $\{h_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to the $\|\cdot\|_{1,\mu}$-seminorm since, for all $l, k \in \mathbb{N}$,

$$\|h_l - h_k\|_{1,\mu} = \int_S |h_l - h_k| \mathrm{d}\mu = \int_S \left| \int_T ((f_l)_p - (f_k)_p) \,\mathrm{d}\nu \right| \mathrm{d}\mu(p)$$
$$\leq \int_S \left(\int_T |(f_l)_p - (f_k)_p| \,\mathrm{d}\nu \right) \mathrm{d}\mu(p) = \|f_l - f_k\|_1$$

and $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy. Thus, by Proposition 3.25, h is integrable and $\{h_n\}_{n\in\mathbb{N}}$ converges to h in the $\|\cdot\|_{1,\mu}$-seminorm. Then,

$$\begin{aligned}\int_{S\times T} f \,\mathrm{d}(\mu\boxtimes\nu) &= \lim_{n\to\infty}\int_{S\times T} f_n \,\mathrm{d}(\mu\boxtimes\nu) = \lim_{n\to\infty}\int_S \left(\int_T (f_n)_p \,\mathrm{d}\nu\right)\mathrm{d}\mu(p)\\ &= \lim_{n\to\infty}\int_S h_n \,\mathrm{d}\mu = \int_S h\,\mathrm{d}\mu = \int_S\left(\int_T f_p\,\mathrm{d}\nu\right)\mathrm{d}\mu(p).\end{aligned}$$

□

Lemma 5.11. *Let $(S,\mathcal{M},\mu)$ and $(T,\mathcal{N},\nu)$ be measure spaces with σ-finite complete measures and $f : S\times T\to\mathbb{K}$ measurable with respect to $(\mathcal{M}\boxtimes\mathcal{N})^*$. Then, for almost all $p\in S$, f_p is measurable with respect to $\mathcal{N}$.*

Proof. By Proposition 2.30, there is a function $g : S\times T\to\mathbb{K}$ that is measurable with respect to $\mathcal{M}\boxtimes\mathcal{N}$ and such that g coincides with f at least outside a set $N\in\mathcal{M}\boxtimes\mathcal{N}$ of measure 0. By Lemma 5.5, g_p is measurable for all $p\in S$. By Lemma 5.7, $\nu(N_p)=0$ for all $p\in S\setminus Y$, where $Y\in\mathcal{N}$ is of measure 0. Let $p\in S\setminus Y$, then g_p coincides with f_p almost everywhere and since $(T,\mathcal{N},\nu)$ is complete f_p must be measurable. □

Theorem 5.12 (Fubini's Theorem, Part 2)**.** *Let $(S,\mathcal{M},\mu)$ and $(T,\mathcal{N},\nu)$ be measure spaces with σ-finite complete measures and $f : S\times T\to\mathbb{K}$ be measurable with respect to $(\mathcal{M}\boxtimes\mathcal{N})^*$. Suppose that $f_p\in\mathcal{L}^1(T,\mathcal{N},\nu)$ for almost all $p\in S$. Moreover suppose that the function*

$$p\mapsto\int_T |f_p|\,\mathrm{d}\nu$$

defined almost everywhere in this way is in $\mathcal{L}^1(S,\mathcal{M},\mu)$. Then, $f\in\mathcal{L}^1(S\times T,(\mathcal{N}\boxtimes\mathcal{M})^*,\mu\boxtimes\nu)$.

Proof. Denote by $X\in\mathcal{M}$ a set of measure 0 such that $f_p\in\mathcal{L}^1(T,\mathcal{N},\nu)$ for $p\in S\setminus X$. By Theorem 2.23 there exists a an increasing sequence $\{f_n\}_{n\in\mathbb{N}}$ of simple functions $f_n : S\times T\to\mathbb{R}^+_0$ with respect to $(\mathcal{M}\boxtimes\mathcal{N})^*$ that converges pointwise to $|f|$. Moreover, because of σ-finiteness the f_n can be chosen to have finite support. (**<u>Exercise</u>.**Explain!) In particular, this implies that each f_n is integrable. Applying Theorem 5.10 to f_n yields a set $N_n\in\mathcal{M}$ of measure 0 such that $(f_n)_p\in\mathcal{L}^1(T,\mathcal{N},\nu)$ for all $p\in S\setminus N_n$. Moreover, it implies that $h_n : S\to\mathbb{R}^+_0$ defined by $h_n(p) := \int_T (f_n)_p\,\mathrm{d}\nu$ for $p\in S\setminus N_n$ and $h_n(p)=0$ otherwise, is integrable. Also it implies,

$$\int_S h_n\,\mathrm{d}\mu = \int_{S\times T} f_n\,\mathrm{d}(\mu\otimes\nu)$$

Let $N := \bigcup_{n\in\mathbb{N}} N_n$. This has measure 0. Note that since $f_n\le f$ for all $n\in\mathbb{N}$ we also have $h_n(p)\le\int_T |f_p|\,\mathrm{d}\nu$ for all $p\in S\setminus\{N\cup X\}$. Putting things together we get for all $n \in \mathbb{N}$

$$\int_{S\times T} f_n \,\mathrm{d}(\mu\otimes\nu) = \int_S h_n \,\mathrm{d}\mu \le \int_S \int_T f_p \,\mathrm{d}\nu$$

Thus, by the Monotone Convergence Theorem 3.26, $f_{n\,n\in\mathbb{N}}$ converges pointwise almost everywhere to an integrable function. But $f_{n\,n\in\mathbb{N}}$ converges pointwise to $|f|$, which is measurable, so $|f|$ must be integrable. Then, by Proposition 3.30, f is integrable. $\square$